PDE of mixed type: The twin challenges of globalization and diversity

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Overview

- PDE of mixed type arise in particular, but interesting physical and geometric contexts

- Regularity of desired (admissible) solutions plays a key role

- The search for type independent techniques increases understanding

- Global information requires overcoming the presence of "diversity"

- The main struggle is for suitable a priori estimates

- Mixed type equations provide an inviting playground for PDE, functional and real analysis techniques

Mixed type PDE: a model class

Consider solutions $u = u(x, y)$ in $\Omega \subset \mathbb{R}^2$ to

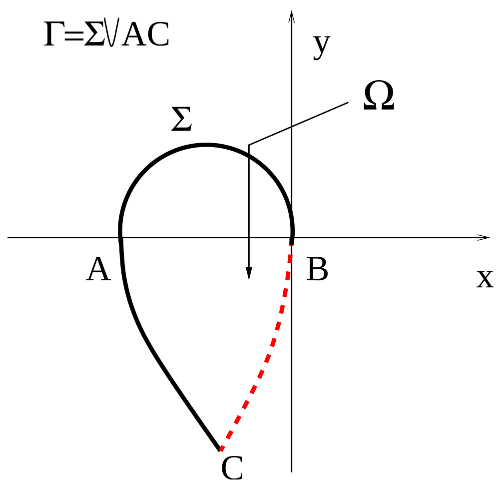
$$K(y)u_{xx} + u_{yy} = f \qquad (1)$$

- $f = f(x, y)$ or $f = f(x, y, u)$
- K is a type change function s.t. $yK(y) > 0$
- An elliptic/hyperbolic equation for $y > 0/y < 0$
- The choice $K(y) = y$ gives the Tricomi operator [Tricomi'23]
- The class with $K = K(x)$ is completely different; Keldysh operator

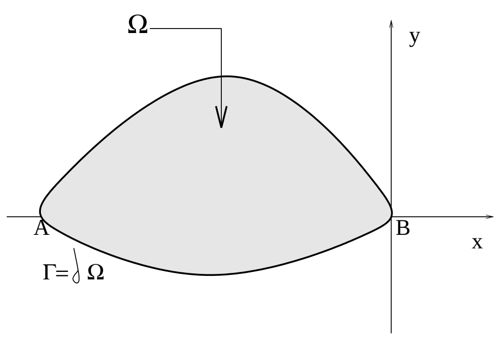
Boundary conditions (BCs): a delicate issue; for example, of Dirichlet type

$$u = g \quad \text{on} \quad \Gamma \subseteq \partial\Omega$$

An "open BC" $\Gamma = AC \cup \Sigma \subset \partial\Omega$ [Tricomi'23]



A "closed BC" $\Gamma = \partial\Omega$ [Morawetz '70]



Applications of mixed type PDE

Mixed type equations like (1) are found in:

Physical problems:

Transonic gas dynamics *
Mechanics: systems of conservation laws
Plasma physics
Non-geometrical optics
String theory

Geometrical problems:

Prescribing curvature: sign change *
Isometric embeddings: sign change* or $n \geq 3$
General relativity: mixed signature metrics
Projective geometry

Physical application: transonic flow

- 2-D irrotational, stationary, compressible, isentropic flow: velocity field (U,V), velocity $q^2 = U^2 + V^2$, density ρ, pressure p

- $\exists$ velocity potential φ s.t. $\nabla\varphi = (U,V)$ and $\exists$ stream function ψ s.t. $\rho(\varphi_x,\varphi_y) = (\psi_y, -\psi_x)$

$$\left(1-\frac{U^2}{c^2}\right)\varphi_{xx} - 2\frac{UV}{c^2} + \left(1-\frac{V^2}{c^2}\right)\varphi_{yy} = 0 \tag{2}$$

with similar equation for ψ

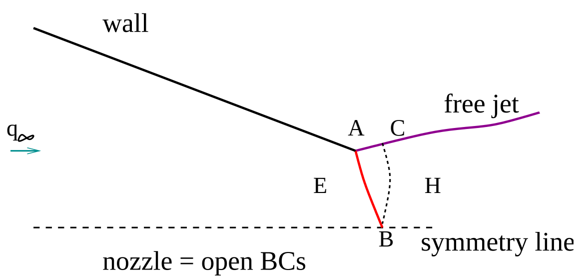
- $c^2 := \dfrac{dp}{d\rho}$ (Ex: $\gamma\rho^{\gamma-1}, \gamma \approx 1.4$ for air)

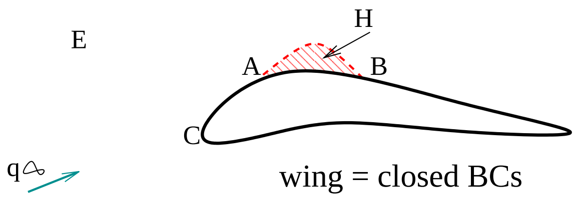
- *Mach number* $M = q/c$

- $M > 1/M < 1$ is supersonic / subsonic flow $\Leftrightarrow$ (2) is hyperbolic / elliptic

Transonic flow: definitions

- Have regions of sub/supersonic flow
- Equations for φ, ψ are quasi-linear (equation type depends on solution)
- Interface sonic line is a free boundary





Boundary conditions: $\partial\varphi/\partial n = 0, \psi = \text{const.}$ along profiles, lines of symmetry and $q = q_\infty$ at infinity

Transonic flow: known results

- Existence theory for subsonic flows [50s - Shiffman, Gilbarg, Bers]. Sonic-subsonic cases [Chen-Dafermos-Slemrod-Wang'07]

- Examples of continuous transonic flows exist [Lighthill'47, Tomatika-Tamada'50, Nieuwland'64], and computational constructions too [Bauer,Garabedian,Jameson,Korn 70s]

- Continuous transonic flows are "unstable" [Morawetz'56-58]. One must expect shocks in general; strength is important (drag)

- Existence and/or uniqueness of stable shock solutions [Xin-Yin, G.Q. Chen-Feldman, E.H. Kim '04-'09]

- Also for transonic small disturbance equation [Cole-Cook'75-95, Keyfitz et al. '98-] or entropy solutions [Chen-Slemrod-Wang'08]

Transonic flow: linearization via hodograph

Consider ψ stream function (adiabatic gas)

- $\psi = \psi(q, \theta)$; $\theta = \tan^{-1}(U/V)$ is flow angle
- $\sigma = -\int_{q_{cr}}^{q} \frac{\rho}{\rho_{cr}} \frac{dt}{t}$ log rescaled speed
- $q_{cr} = \left(\frac{2}{1+\gamma}\right)^{1/2}, \rho_{cr} = \left(1 - \frac{\gamma - 1}{2} q_{cr}^2\right)^{1/(\gamma-1)}$

$$K(\sigma)\tilde{\psi}_{\theta\theta} + \tilde{\psi}_{\sigma\sigma} = 0; \quad \tilde{\psi} = \frac{\psi}{\rho_{cr} q_{cr}} \qquad (3)$$

- Perturbation of a known potential $\tilde{\varphi} = \delta\varphi$ solves an equation like (3) ($\varphi = \varphi_0 + \delta\varphi$)

N.B. $K(\sigma) \sim \sigma$ for $\sigma \to 0$

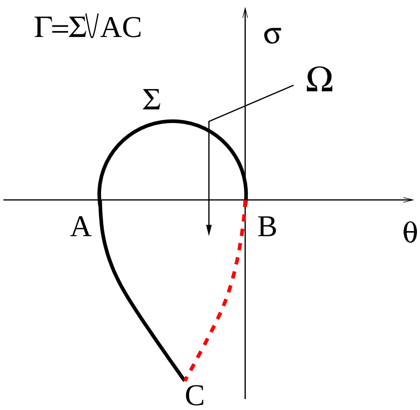
BCs in the hodograph plane:

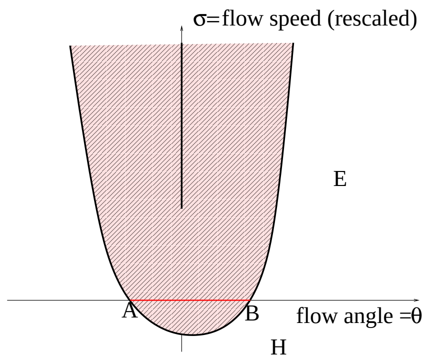
On profiles, symmetry lines

- $\tilde{\psi} = \text{const.}$

- $\tilde{\varphi}_\nu := (K\tilde{\varphi}_\theta, \tilde{\varphi}_\sigma) = 0$ (conormal derivative)

Hodograph images (nozzles/wings)





Geometric application: surfaces w/ sign change in curvature

Problem 1: Find a graph $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ with prescribed Gauss curvature $K(x) = K(x_1, x_2)$;

- u solves a Monge-Ampere equation

$$\det\left(D^2 u\right) = K(x)(1 + |\nabla u|^2)^2 \qquad (4)$$

- eq. (4) is $D(rt - s^2) + Ar + Bs + Ct + E = 0$; $(x, p, q, r, s, t) = (x, u, \partial_1 u, \partial_2 u, \partial_{11} u, \partial_{12} u, \partial_{22} u)$; $A, \ldots, E$ smooth functions of (x, p, q, r)

- Sign of $B^2 - 4AC + 4ED = -4K(1 + |\nabla u|^2)^2$ determines type; have elliptic/hyperbolic for $K(x) > 0/K(x) < 0$

Prescribed Gauss curvature: known results

- Local existence near $P \in \Omega$: K real analytic or $K(P) > 0/K(P) < 0$ [Janet, Cartan'20s, Hartman-Winter'50s, Poznyak'66-'73, Jacobowitz'82]

- Global existence: $K > 0$ and Ω convex [Caffarelli- Nirenberg-Spruck'84]

- Local existence: $K \geq 0$ in a neighborhood of P or $K(P) = 0, \nabla K(P) \neq 0$ [C.S. Lin'95] or K vanishes of finite order along a smooth curve [Khuri'07]

- K very regular (C^6 for Lin, C^{58} for Khuri)

- Counterexamples: $\exists$ $\{A, B, C, E\}$ w/ E changing sign, s.t. $\nexists C^3$ local solutions [Khuri'07]

Problem 2: Given a surface (M^2, g), does there exist a local isometric embedding into $\mathbb{R}^3$; i.e. for $P \in M^2$ find a neighborhood $\Omega \subset \mathbb{R}^2$ and a map $\mathbf{r} = (X_1, X_2, X_3) : \Omega \subset \mathbb{R}^2 \hookrightarrow \mathbb{R}^3$ s.t. $d\mathbf{r} \cdot d\mathbf{r} = g$ holds on Ω

- in coordinates (x_1, x_2) with $g = g_{ij} dx_i dx_j$, one solves for (X_1, X_2, X_3) in:

$$g_{ij} = \partial_i X_k \partial_j X_k; \quad i, j = 1, 2$$

- each $u = X_k$ solves the Darboux equation

$$\det(D_g^2 u) = K|g| \left(1 - |\nabla_g u|^2\right) \qquad (5)$$

where $|g| = \det[g_{ij}]$ and

$$[D_g^2 u]_{ij} = \partial_{ij} u - \Gamma_{ij}^k \partial_k u; \quad |\nabla_g u|^2 = g^{ij} \partial_i u \partial_j u$$

$$[g^{ij}] = [g_{ij}]^{-1}; \quad \Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ij})$$

- look for u with $|\nabla_g u|^2 < 1$ and (5) is again elliptic/hyperbolic if $K > 0 / K < 0$

Local isometric embedding: known results

- Conjecture Schlaefli 1873: always possible if metric is smooth

- K real analytic or $K(P) > 0/K(P) < 0$ [Janet, Cartan'20s, Hartman-Winter'50s, Poznyak'66-'73, Jacobowitz'82]

- Counterexamples w/ $K \geq 0$ or K changing sign: $\exists\, g \in C^{2,1}$ with no $\mathbf{r} \in C^2$ [Pogorelov'71]

- Positive results in certain degenerate and sign changing cases [Lin'85-86, Nakamura'97, Han-Hong-Lin'02, Han'05, Khuri'07]

Geometric problems: local existence

Nash-Moser implicit function approach: (Darboux eq. (5) with K vanishing to order m along a curve through $(x_1, x_2) = (0, 0)$)

- Rescale $(x_1, x_2) = (\epsilon^{m+1}x, \epsilon^2 y)$ w/ $\epsilon > 0$; seek u as a small perturbation w of presumed approximate solution u_0; the perturbation w satisfies

$$y^m w_{xx} + w_{yy} + \epsilon f(\epsilon, x, y, Dw, D^2 w) = 0 \tag{6}$$

- Lots of estimates to have existence and regularity theory for the linearization of (6) about w and tame dependence w.r.t. w of the solutions to the linearized problem

- Estimates in L^2 based Sobolev spaces and immersions into C^k; large regularity loss to block shrinkage to a point.

Challenges: mixed type PDE

- Global existence for Prescribed Gauss Curvature Equation or Darboux Equation with curvature changing sign

- Suitable global estimates with respect to L^2 based Sobolev spaces* or other (better?) norms

- Understand better the structure of transonic flow near the sonic line

- Construct candidates for continuous transonic flows in the hodograph plane*

- Existence of nontrivial solutions to the Dirichlet problem for linear/semi-linear equations such as (1)*

Obtaining global information: techniques

For equations of the form (1)

$$K(x)u_{xx} + u_{yy} = f(x, y, u)$$

together with open and/or closed BCs:

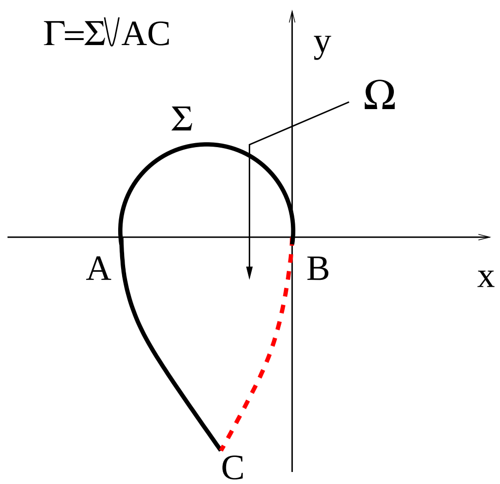
- Suitable maximum principles w/ open BCs: classical solutions [Germain-Bader, Agmon-Nirenberg-Protter '50s] and weak solutions [Lupo-P.'00]

- Energy identities and inequalities (multiplier methods): [Protter'53, Morawetz'58, Didenko'73, Lupo - P.'03-'07]

- Conservation laws associated to invariances in (1): [Morawetz'58, Lupo - P.'05-'07]

Maximum principles

Consider Ω a Tricomi domain w/ $\partial\Omega = \Sigma \cup AC \cup BC$ and $L = K(y)\partial_{xx} + \partial_{yy}$ with $K \in C^2$ s.t. $5(K')^2 \geq 4KK''$



Theorem: [Agmon-Nirenberg-Protter, CPAM'53] *If $u \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus \{A, B\}) \cap C^0(\overline{\Omega})$ satisfies $Lu \geq 0$ and u increasing from C to A, then $\max_{\overline{\Omega}} u = \max_{\overline{\Sigma}} u$.*

Why? Integral identity along characteristic segment $[P, Q]_+$ from $P \in AC$ to $Q \in \Omega^-$ shows $D_- u(Q) > 0$. Then Hopf lemma on AB plus elliptic theory.

Consequence: The Dirichlet problem on Ω is overdetermined for classical solutions

In fact: $Lu = 0$ in Ω plus $u = 0$ on $AC \cup \Sigma$ implies $u \equiv 0$ on $\overline{\Omega}$, so u on BC is already determined.

Weak solutions: [Lupo-P., CCM'00] For normal domains Ω and $K(y) = y$ have maximum principle compatible with an existence theory in $H^1_{AC\cup\sigma}(\Omega; K)$ the completion of $\{u \in C^1(\overline{\Omega}) : u = 0 \text{ near } AC \cup \Sigma, \int_\Omega \left(|K|u_x^2 + u_y^2 + u^2\right) < \infty\}$

- For $f \geq 0$ in $L^2(\Omega)$, $Lu \leq 0 \Rightarrow u \geq 0$

- Get a principal eigenvalue for $-L$ via Krein-Rutman argument [Lupo-P., EJDE'00]

Multiplier methods

Idea: For $L = K\partial_{xx} + \partial_{yy}$, select φ so that manipulation of $\int_\Omega \varphi L u$ or $\int_\Omega \varphi L^t u$ yields a potentially useful energy identity or inequality

1. Differential multiplier: pick $\varphi = Mu := au + bu_x + cu_y$ with (a, b, c) TBA [Freidrichs]

- [Protter, JRMA'53]-uniqueness of classical solutions by using $u \in C^2(\overline{\Omega})$ solution to the homogeneous BVP $Lu = 0, u_{|\Gamma} = 0$

- [Berezanskii, Dolk.'53]-existence of ultra-weak solutions $u \in L^2(\Omega)$, uniqueness of strong solutions by estimating w/ $u \in C_0^2(\Omega)$

2. Matrix multiplier: Reduce PDE to 1st order system

$$\mathcal{L} = \begin{pmatrix} K\partial_x & \partial_y \\ \partial_y & -\partial_x \end{pmatrix} \text{ then pick } \mathcal{M} = \begin{pmatrix} b & c \\ -Kc & b \end{pmatrix}$$

- [Morawetz, CPAM'58]-existence/uniqueness of weak solutions (open BC)

- [Morawetz, CPAM'70]-also for Dirichlet problem w/ very special domain and $K(y) = y$

3. Integral multiplier: pick $\varphi = Iu$ solution to $M\varphi = a\varphi + b\varphi_x + c\varphi_y = u, \varphi_{|\Gamma} = 0$ and estimate w/ $u \in C_0^2(\Omega)$

- [Didenko, Ukr.MJ'73, Lupo-P., CCM'00]-existence/uniqueness of weak solutions (open BC)

- [Lupo-Morawetz-P., CPAM'07]-also for Dirichlet problem w/ suitably starshaped domains and $K(y) = y|y|^{m-1}$

Weak well-posedness: Dirichlet problem

Look for weak solution $u \in H^1_0(\Omega; K; \epsilon)$; i.e.

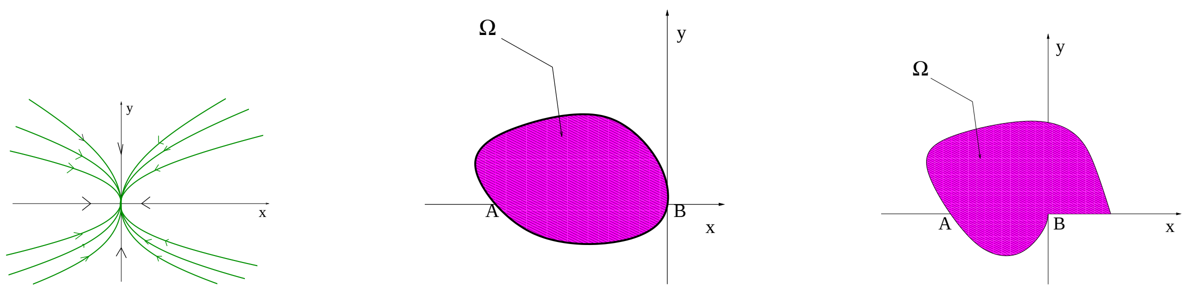
$$\int_\Omega K u_x \varphi_x + u_y \varphi_y = -\int_\Omega f\varphi, \quad \forall \varphi \in C_0^\infty(\Omega)$$

where $H^1_0(\Omega; K; \epsilon)$ is the completion of $C_0^\infty(\Omega)$ w.r.t $[\int_\Omega \left(|K|^{1+\epsilon} u_x^2 + u_y^2\right)]^{1/2}$ and $\epsilon > 0$.

Theorem [LMP], [LP]: *For suitable Ω, K, given $\epsilon > 0$ and $f \in L^2(\Omega, K^{-1}dxdy)$ there exists a unique weak solution in $H^1_0(\Omega; K; \epsilon)$.*

Suitable K: power type $K(y) = y|y|^{m-1}, m > 0$ or in a general class including adiabatic gas K

Suitable Ω: V-star-shaped w.r.t. $B(0,0) \in \partial\Omega$ w/ $V = -(b, c)$; in power case $b = (m+2)x, c = \mu^\pm y$ w/ $\mu^+ = 2, \mu^- = 1$



Why V-star-shaped? Ensures that the auxiliary Cauchy problem makes sense:

$$M\varphi = u \quad \text{in} \quad \Omega; \quad \varphi = 0 \quad \text{on} \quad \partial\Omega \setminus (0,0)$$

Rest of proof:

- Show $\varphi \in H_0^1(\Omega; K, \epsilon), \epsilon > 0$

- Get global energy estimate: $\forall u \in C_0^\infty(\Omega)$

$$||u||_{L^2(\Omega, K\, dxdy)} \leq C||Lu||_{H^{-1}(\Omega; K, \epsilon)}$$

- Functional analysis (Lax representation thm.)

Choice of (a, b, c): in $M = a + b\partial_x + c\partial_y$

- $a < 0$ to get decay of φ along flow lines of $V = -(b, c)$

- ML^t is a positive operator

- V is almost the infinitesimal generator of a dilation invariance for $L = L^t$

Non-existence for semilinear problems

Point: $L = y|y|^{m-1}\partial_{xx} + \partial_{yy}, m > 0$ has a dilation invariance with infinitesimal generator $V = ((m+2)x, 2y)$

Theorem: [Lupo-P., CPAM'03] *There are no nontrivial classical solutions to the semilinar Tricomi problem*

$$Lu = Cu|u|^{p-2} \quad \text{in} \quad \Omega; \quad u = 0 \quad \text{on} \quad AC \cup \Sigma$$

if Ω is V-star-shaped and $p > 2^*(m)$.

N.B. $2^*(m) = 2(m+4)/m = 2N/(N-2)$ is the critical Sobolev exponent corresponding to the homogeneous dimension $N = (m+4)/2$.

Why? Pohožaev-type argument combined w/ sharp Hardy-Sobolev inequality on BC (where no boundary condition is imposed)

Invariances for mixed type PDE

Question: What are the invariances for a mixed type operator $L = K(y)\Delta_x + \partial_{yy}$?

Answer:

- In general, just translations in x - used for uniqueness theorems [Morawetz, PRSL'58] (maximum principles for the potential function of the associated conservation law)

- For $K(y) = y|y|^{m-1}$, the complete symmetry group is generated by translations and rotations in $x \in \mathbb{R}^n$, dilations, inversions [Lupo-P., DMJ'05] - various applications

- This symmetry group is a conformal group of transformations w.r.t a singular metric of mixed Riemannian-Lorenztian signature [P., AMPA'06]